A property of isometric mappings between dual polar spaces of type $DQ(2n, \mathbb{K})$

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Abstract

Let f be an isometric embedding of the dual polar space $\Delta = DQ(2n, \mathbb{K})$ into $\Delta' = DQ(2n, \mathbb{K}')$. Let P denote the point-set of Δ and let $e' : \Delta' \to \Sigma' \cong \mathrm{PG}(2^n - 1, \mathbb{K}')$ denote the spin-embedding of Δ'. We show that for every locally singular hyperplane H of Δ, there exists a unique locally singular hyperplane H' of Δ' such that $f(H) = f(P) \cap H'$. We use this to show that there exists a subgeometry $\Sigma \cong \mathrm{PG}(2^n - 1, \mathbb{K})$ of Σ' such that: (i) $e' \circ f(x) \in \Sigma$ for every point x of Δ; (ii) $e := e' \circ f$ defines a full embedding of Δ into Σ, which is isomorphic to the spin-embedding of Δ.

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1 Introduction

1.1 Basic definitions

Let Π be a nondegenerate polar space of rank $n \geq 2$. With Π there is associated a point-line geometry Δ whose points are the maximal singular subspaces of Π, whose lines are the next-to-maximal singular subspaces of Π and whose incidence relation is reverse containment. The geometry Δ is

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called a *dual polar space* (Cameron [2]). There exists a bijective correspondence between the nonempty convex subspaces of Δ and the possibly empty singular subspaces of Π: if α is a singular subspace of Π, then the set of all maximal singular subspaces containing α is a convex subspace of Δ. The maximal distance (in the collinearity graph) between two points of a convex subspace A of Δ is called the *diameter* of A and is denoted as $\mathrm{diam}(A)$. The convex subspaces of diameter 2, 3, respectively $n-1$, of Δ are called the *quads*, *hexes*, respectively *maxes*, of Δ. The convex subspaces through a given point x of Δ define an $(n-1)$-dimensional projective space which we will denote by $Res_\Delta(x)$.

For every two points x and y of Δ, $\mathrm{d}(x,y)$ denotes the distance between x and y in the collinearity graph of Δ and $\langle x,y\rangle$ denotes the smallest convex subspace containing x and y. We have $\mathrm{diam}\langle x,y\rangle = \mathrm{d}(x,y)$. More generally, if $*_1, *_2, \ldots, *_k$ are $k \geq 1$ objects of Δ (like points or convex subspaces), then $\langle *_1, *_2, \ldots, *_k\rangle$ denotes the smallest convex subspace of Δ containing the objects $*_1, *_2, \ldots, *_k$. If A and B are two nonempty sets of points of Δ, then $\mathrm{d}(A,B)$ denotes the smallest distance between a point of A and a point of B. If x is a point of Δ and if $i \in \mathbb{N}$, then $\Delta_i(x)$ denotes the set of points at distance i from x. We define $x^\perp := \Delta_0(x) \cup \Delta_1(x)$. For every point x and every convex subspace A of Δ, there exists a unique point $\pi_A(x)$ in A nearest to x and $\mathrm{d}(x,y) = \mathrm{d}(x,\pi_A(x)) + \mathrm{d}(\pi_A(x),y)$ for every point y of A. We call $\pi_A(x)$ the *projection* of x onto A. If A and B are two convex subspaces of Δ, then we define

$$\mathrm{ch}(A,B) := (\mathrm{diam}(A), \mathrm{diam}(B), \mathrm{d}(A,B), \mathrm{diam}\langle A,B\rangle).$$

$\mathrm{ch}(A,B)$ is called the *characteristic* of (A,B). The characteristic of (A,B) describes the mutual position of A and B.

In this paper, we are mainly interested in the dual polar space $DQ(2n,\mathbb{K})$ which is associated with a nonsingular quadric of Witt-index $n \geq 2$ in $\mathrm{PG}(2n,\mathbb{K})$.

A *hyperplane* of a point-line geometry is a proper subspace meeting each line. Suppose H is a hyperplane of a thick dual polar space Δ of rank $n \geq 2$. By Shult [9, Lemma 6.1], we then know that H is a maximal subspace of Δ. A point x of H is called *deep (with respect to H)* if $x^\perp \subseteq H$. If H consists of all points of Δ at non-maximal distance from a given point y, then H is called the *singular hyperplane* of Δ with deepest point y. One of the following cases occurs for a quad Q of Δ: (i) $Q \subseteq H$; (ii) $Q \cap H = x^\perp \cap Q$ for a certain point $x \in Q$; (iii) $Q \cap H$ is an ovoid of Q; (iv) $Q \cap H$ is a subquadrangle of Q. If only cases (i) or (ii) occur, then H is called *locally singular*. A set $\mathcal{W}$ of hyperplanes of a dual polar space Δ is called a *pencil of hyperplanes* if every point of Δ is contained in either one or all hyperplanes of $\mathcal{W}$.

A *full embedding* of a point-line geometry $\mathcal{S}$ into a projective space Σ is an injective mapping e from the point-set P of $\mathcal{S}$ to the point-set of Σ satisfying (i) $\langle e(P) \rangle = \Sigma$ and (ii) $e(L)$ is a line of Σ for every line L of $\mathcal{S}$. If e is a full embedding of $\mathcal{S}$, then for every hyperplane α of Σ, the set $e^{-1}(e(P) \cap \alpha)$ is a hyperplane of $\mathcal{S}$. We say that the hyperplane $e^{-1}(e(P) \cap \alpha)$ *arises from the embedding* e. The dual polar space $DQ(2n, \mathbb{K})$, $n \geq 2$, has a nice full projective embedding into $\mathrm{PG}(2^n - 1, \mathbb{K})$, which is called the *spin-embedding* of $DQ(2n, \mathbb{K})$. We refer to Chevalley [4] or Buekenhout and Cameron [1] for definitions and background information on the topic of spin-embeddings.

1.2 The main results

Definition. Let Δ and Δ' be two dual polar spaces with respective point-sets P and P'. We denote the distance function in Δ and Δ' respectively by $\mathrm{d}(\cdot, \cdot)$ and $\mathrm{d}'(\cdot, \cdot)$. An *isometric embedding* of Δ into Δ' is a map $f : P \rightarrow P'$ satisfying

$$\mathrm{d}'(f(x), f(y)) = \mathrm{d}(x, y)$$

for all points x and y of P.

Example. Let $n \in \mathbb{N} \setminus \{0, 1\}$ and let $\mathbb{K}$ and $\mathbb{K}'$ be fields such that $\mathbb{K}$ is a subfield of $\mathbb{K}'$. Every point of the projective space $\mathrm{PG}(2n, \mathbb{K})$ can be regarded as a point of the projective space $\mathrm{PG}(2n, \mathbb{K}')$. For every subspace α of $\mathrm{PG}(2n, \mathbb{K})$, let $f(\alpha)$ denote the subspace of $\mathrm{PG}(2n, \mathbb{K}')$ generated by all points of α. The equation $X_0^2 + X_1 X_2 + \cdots + X_{2n-1} X_{2n} = 0$ defines a quadric $Q(2n, \mathbb{K})$ of Witt-index n in $\mathrm{PG}(2n, \mathbb{K})$ and a quadric $Q(2n, \mathbb{K}')$ of Witt-index n in $\mathrm{PG}(2n, \mathbb{K}')$. The map f restricted to the set of generators (= maximal singular subspaces) of $Q(2n, \mathbb{K})$ defines an isometric embedding of $DQ(2n, \mathbb{K})$ into $DQ(2n, \mathbb{K}')$.

In Section 2, we will study isometric embeddings between general dual polar spaces. We also notice there that if there exists an isometric embedding of $DQ(2n, \mathbb{K})$ into $DQ(2n', \mathbb{K}')$, $3 \leq n \leq n'$, then $\mathbb{K}$ is isomorphic to a subfield of $\mathbb{K}'$.

In Section 3, we will derive some properties of locally singular hyperplanes of $DQ(2n, \mathbb{K})$. We will use these properties in Section 4 to prove the following result:

Theorem 1.1 (Section 4) *Let f be an isometric embedding of the dual polar space $DQ(2n, \mathbb{K})$ into the dual polar space $DQ(2n, \mathbb{K}')$, $n \geq 2$. Let P denote the point-set of $DQ(2n, \mathbb{K})$. Then for every locally singular hyperplane H of $DQ(2n, \mathbb{K})$, there exists a unique locally singular hyperplane H' of $DQ(2n, \mathbb{K}')$ such that $f(H) = f(P) \cap H'$.*

Theorem 1.1 will be used in [7] to show that certain classes of hyperplanes of dual polar spaces arise from embedding. Theorem 1.1 will be used in Section 5 to show the following.

Theorem 1.2 (Section 5) *Let f be an isometric embedding of the dual polar space $\Delta = DQ(2n, \mathbb{K})$ into the dual polar space $\Delta' = DQ(2n, \mathbb{K}')$, $n \geq 2$. Let $e' : \Delta' \to \Sigma' \cong \mathrm{PG}(2^n - 1, \mathbb{K}')$ denote the spin-embedding of Δ'. Then there exists a subgeometry $\Sigma \cong \mathrm{PG}(2^n - 1, \mathbb{K})$ of Σ' such that the following holds:*

- *(i) $e' \circ f(x) \in \Sigma$ for every point x of Δ;*
- *(ii) $e := e' \circ f$ defines a full embedding of Δ into Σ, which is isomorphic to the spin-embedding of Δ.*

2 Properties of isometric embeddings

Let Δ and Δ' be two dual polar spaces with respective point sets P and P' and suppose that $f : P \to P'$ is an isometric embedding of Δ into Δ'.

Proposition 2.1 *For every convex subspace A of Δ, there exists a unique convex subspace A_f of Δ' satisfying*

- *(1) A and A_f have the same diameter;*
- *(2) $f(x) \in A_f$ for every point $x \in A$.*

Proof. (i) Obviously, the proposition holds if $\mathrm{diam}(A) = 0$ ($A_f = f(A)$ in this case).

(ii) Suppose $\mathrm{diam}(A) = 1$. So, A is a line. Let x and y denote two distinct points of A. If A_f is a convex subspace of Δ' satisfying (1) and (2), then A_f necessarily coincides with the unique line B through $f(x)$ and $f(y)$. Now, if z is a point of $A \setminus \{x, y\}$, then $f(z) \in A_f$ since $\mathrm{d}'(f(z), f(y)) = \mathrm{d}(z, y) = 1$ and $\mathrm{d}'(f(z), f(x)) = \mathrm{d}(z, x) = 1$. This shows that B is indeed the unique convex subspace satisfying (1) and (2).

(iii) Suppose $\mathrm{diam}(A) \geq 2$. Let x and y denote two points of A at distance $\mathrm{diam}(A)$ from each other. If A_f is a convex subspace of Δ' satisfying properties (1) and (2), then since $\mathrm{d}'(f(x), f(y)) = \mathrm{d}(x, y) = \mathrm{diam}(A)$, A_f necessarily coincides with the smallest convex subspace B of Δ' containing $f(x)$ and $f(y)$. Now, f satisfies the following properties:

- f maps every line of Δ into a line of Δ' (see (ii));
- f maps a shortest path in Δ to a shortest path in Δ'.

Hence, f maps the smallest convex subspace through x and y into the smallest convex subspace of Δ' through $f(x)$ and $f(y)$. In other words, $f(A) \subseteq B$. So, the convex subspace B indeed satisfies properties (1) and (2) of the proposition. ■

Corollary 2.2 *There exists a unique convex subspace Δ'' of Δ' satisfying the following properties:*

(i) $\mathrm{diam}(\Delta'') = \mathrm{diam}(\Delta)$*;*

(ii) $f(x) \in \Delta''$ *for every point x of Δ.*

Proposition 2.3 *If x is a point of Δ and if A is a convex subspace of Δ, then $\pi_{A_f}(f(x)) = f(\pi_A(x))$.*

Proof. Let y be a point of A at distance $\mathrm{diam}(A)$ from $\pi_A(x)$. By the proof of Proposition 2.1, $A_f = \langle f(\pi_A(x)), f(y) \rangle$. We have

$$\begin{aligned} \mathrm{d}'(f(x), f(y)) &= \mathrm{d}(x, y) \\ &= \mathrm{d}(x, \pi_A(x)) + \mathrm{d}(\pi_A(x), y) \\ &= \mathrm{d}'(f(x), f(\pi_A(x))) + \mathrm{diam}(A). \end{aligned} \tag{1}$$

From

$$\mathrm{d}'(f(x), f(\pi_A(x))) \geq \mathrm{d}'(f(x), \pi_{A_f}(f(x)))$$

and

$$\mathrm{diam}(A) = \mathrm{diam}(A_f) \geq \mathrm{d}'(\pi_{A_f}(f(x)), f(y)),$$

it follows that

$$\begin{aligned} \mathrm{d}'(f(x), f(\pi_A(x))) + \mathrm{diam}(A) &\geq \mathrm{d}'(f(x), \pi_{A_f}(f(x))) + \mathrm{d}'(\pi_{A_f}(f(x)), f(y)) \\ &= \mathrm{d}'(f(x), f(y)). \qquad (2) \end{aligned}$$

By equations (1) and (2), $\mathrm{d}'(f(x), f(\pi_A(x))) = \mathrm{d}'(f(x), \pi_{A_f}(f(x)))$. Hence, $f(\pi_A(x)) = \pi_{A_f}(f(x))$. ∎

Proposition 2.4 *If A and B are two convex subspaces of Δ, then* $\mathrm{ch}(A, B) = \mathrm{ch}(A_f, B_f)$.

Proof. Obviously, $\mathrm{diam}(A) = \mathrm{diam}(A_f)$ and $\mathrm{diam}(B) = \mathrm{diam}(B_f)$.

We will now show that $\mathrm{d}(A, B) = \mathrm{d}'(A_f, B_f)$. Let x and y be points of A and B, respectively, such that $\mathrm{d}(x, y) = \mathrm{d}(A, B)$. Then $y = \pi_B(x)$ and $x = \pi_A(y)$. By Proposition 2.3, $\pi_{B_f}(f(x)) = f(\pi_B(x)) = f(y)$ and $\pi_{A_f}(f(y)) = f(\pi_A(y)) = f(x)$. Now, let x^* and y^* be points of A_f and B_f, respectively, such that $\mathrm{d}'(x^*, y^*) = \mathrm{d}'(A_f, B_f)$. Then $y^* = \pi_{B_f}(x^*)$ and $x^* = \pi_{A_f}(y^*)$. Without loss of generality, we may suppose that

$$\mathrm{d}'(f(y), x^*) \geq \mathrm{d}'(f(x), y^*). \qquad (3)$$

Now,

$$\begin{aligned} \mathrm{d}'(f(x), y^*) &= \mathrm{d}'(f(x), \pi_{B_f}(f(x))) + \mathrm{d}'(\pi_{B_f}(f(x)), y^*) \\ &= \mathrm{d}'(f(x), f(y)) + \mathrm{d}'(f(y), y^*), \qquad (4) \end{aligned}$$

and

$$\begin{aligned} \mathrm{d}'(f(y), x^*) &= \mathrm{d}'(x^*, \pi_{B_f}(x^*)) + \mathrm{d}'(\pi_{B_f}(x^*), f(y)) \\ &= \mathrm{d}'(x^*, y^*) + \mathrm{d}(y^*, f(y)). \qquad (5) \end{aligned}$$

By (3), (4) and (5),

$$d'(A_f, B_f) = \mathrm{d}'(x^*, y^*) \geq \mathrm{d}'(f(x), f(y)) \geq \mathrm{d}'(A_f, B_f).$$

Hence,

$$\mathrm{d}'(A_f, B_f) = \mathrm{d}'(x^*, y^*) = \mathrm{d}'(f(x), f(y)) = \mathrm{d}(x, y) = \mathrm{d}(A, B).$$

We will now show that $\mathrm{diam}\langle A, B\rangle = \mathrm{diam}\langle A_f, B_f\rangle$. Choose $x \in A$ and $y \in B$ such that $\mathrm{d}(x, y)$ is maximal. Then y lies at maximal distance (i.e. distance $\mathrm{diam}(B)$) from $\pi_B(x)$. Since $\pi_B(x)$ lies on a shortest path between x and y, $\pi_B(x) \in \langle x, y\rangle$ and hence $B = \langle \pi_B(x), y\rangle \subseteq \langle x, y\rangle$. In a similar way one shows that $A \subseteq \langle x, y\rangle$. It follows that $\langle A, B\rangle = \langle x, y\rangle$ and $\mathrm{diam}\langle A, B\rangle = \mathrm{d}(x, y)$.

Now, since $\pi_B(x)$ is on a shortest path between x and y, $f(\pi_B(x))$ is on a shortest path between $f(x)$ and $f(y)$ and hence $B_f = \langle f(\pi_B(x)), f(y)\rangle \subseteq \langle f(x), f(y)\rangle$. In a similar way, one shows that $A_f \subseteq \langle f(x), f(y)\rangle$. So, $\langle A_f, B_f\rangle = \langle f(x), f(y)\rangle$ and $\mathrm{diam}\langle A_f, B_f\rangle = \mathrm{d}'(f(x), f(y)) = \mathrm{d}(x, y) = \mathrm{diam}\langle A, B\rangle$. ■

Proposition 2.5 *If f is an isometric embedding of $\Delta = DQ(2n, \mathbb{K})$ into $\Delta' = DQ(2n', \mathbb{K}')$, $3 \leq n \leq n'$, then $\mathbb{K}$ is isomorphic to a subfield of $\mathbb{K}'$.*

Proof. Let Δ'' be the convex subspace of Δ' as defined in Corollary 2.2. If x is a point of Δ, then $Res_\Delta(x) \cong \mathrm{PG}(n-1, \mathbb{K})$ and $Res_{\Delta''}(f(x)) \cong \mathrm{PG}(n-1, \mathbb{K}')$. By Proposition 2.4, there exists a subgeometry $\Sigma \cong \mathrm{PG}(n-1, \mathbb{K})$ in $\mathrm{PG}(n-1, \mathbb{K}')$ which generates the whole space $\mathrm{PG}(n-1, \mathbb{K}')$. This is only possible when $\mathbb{K}$ is isomorphic to a subfield of $\mathbb{K}'$. ■

3 Properties of locally singular hyperplanes

In this section, Δ denotes the dual polar space $DQ(2n, \mathbb{K})$, $n \geq 2$, and $e : \Delta \to \Sigma = \mathrm{PG}(2^n - 1, \mathbb{K})$ denotes the spin-embedding of Δ. We denote the point-set of Δ by P.

Proposition 3.1 ([5]; [10]) *The locally singular hyperplanes of Δ are precisely the hyperplanes of Δ which arise from the embedding e.*

If H is a locally singular hyperplane of Δ arising from the hyperplane α of Σ, then $\alpha = \langle e(H)\rangle$, since H is a maximal subspace of Δ. So, there exists a bijective correspondence between the locally singular hyperplanes of Δ and the hyperplanes of Σ.

Lemma 3.2 *If H is a locally singular hyperplane of Δ, then H cannot contain two disjoint maxes.*

Proof. Suppose the contrary and let M_1 and M_2 be two disjoint maxes contained in H. Let x denote an arbitrary point of Δ not contained in $M_1 \cup M_2$. If x, $\pi_{M_1}(x)$ and $\pi_{M_2}(x)$ are contained in a line, then $x \in H$, since $\pi_{M_1}(x), \pi_{M_2}(x) \in H$. Suppose x, $\pi_{M_1}(x)$ and $\pi_{M_2}(x)$ are not contained in a line. Then $Q := \langle x, \pi_{M_1}(x), \pi_{M_2}(x) \rangle$ is a quad. Since $Q \cap M_1$ and $Q \cap M_2$ are lines contained in H, Q itself is also contained in H (recall that H is locally singular). In particular, x belongs to H.

It follows that every point of Δ is contained in H. This is impossible since H is a proper subspace of Δ. ■

Lemma 3.3 *Let H_1 and H_2 be two distinct locally singular hyperplanes of Δ, then there exists a point x in Δ not contained in $H_1 \cup H_2$.*

Proof. Let α_i, $i \in \{1, 2\}$, denote the hyperplane of Σ giving rise to H_i. Then $\alpha_1 \neq \alpha_2$ and hence there exists a hyperplane α of Σ through $\alpha_1 \cap \alpha_2$ distinct from α_1 and α_2. Put $H := e^{-1}(e(P) \cap \alpha)$. Then $H_1 \cap H_2 \subseteq H$. Since H, H_1 and H_2 are maximal subspaces, $H_1 \cap H_2$ is not a maximal subspace and there exists a point $x \in H \setminus (H_1 \cap H_2)$. Obviously, $x \notin H_1 \cup H_2$. ■

Lemma 3.4 *Let M_1 and M_2 be two disjoint maxes, let H_i, $i \in \{1, 2\}$, denote a locally singular hyperplane of M_i and let L be a line of Δ such that $L \cap M_i$ is a singleton $\{x_i\}$ not contained in H_i ($i \in \{1, 2\}$). Then for every point x of L, there exists a unique locally singular hyperplane of Δ containing $H_1 \cup H_2 \cup \{x\}$.*

Proof. Put $\Sigma_i := \langle e(M_i) \rangle$, $i \in \{1, 2\}$. By De Bruyn [6, Theorem 1.1], $\Sigma_1 \cap \Sigma_2 = \emptyset$ and $\langle \Sigma_1, \Sigma_2 \rangle = \Sigma$. Moreover, e induces a full embedding e_i of M_i into Σ_i ($i \in \{1, 2\}$) which is isomorphic to the spin-embedding of $M_i \cong DQ(2n-2, \mathbb{K})$. (If $n = 2$, then e_i is just the embedding of the line M_i into $\mathrm{PG}(1, \mathbb{K})$.) Since H_i is a locally singular hyperplane of M_i, $\alpha_i := \langle e_i(H_i) \rangle = \langle e(H_i) \rangle$ is a hyperplane of Σ_i by Proposition 3.1. Notice that $\dim(\alpha_1) = \dim(\alpha_2) = 2^{n-1} - 2$.

Claim. The space $\langle \alpha_1, \alpha_2 \rangle$ is disjoint from $e(L)$.
Proof. Suppose the contrary. Let y be a point of L such that $e(y) \in \langle \alpha_1, \alpha_2 \rangle$. Without loss of generality, we may suppose that $y \neq x_1$. The space $\langle \alpha_1, \alpha_2, e(x_1) \rangle$ contains $e(H_1)$ and $e(x_1)$ and hence also every point $e(z)$, $z \in M_1$, since H_1 is a maximal subspace of M_1. Hence, $\Sigma_1 \subseteq \langle \alpha_1, \alpha_2, e(x_1) \rangle$. Now, since $e(y) \in \langle \alpha_1, \alpha_2, e(x_1) \rangle$ and $y \neq x_1$, $e(z) \in \langle \alpha_1, \alpha_2, e(x_1) \rangle$ for every point z of the line $L = x_1 x_2$. In particular, $e(x_2) \in \langle \alpha_1, \alpha_2, e(x_1) \rangle$. Since

$\langle \alpha_1, \alpha_2, e(x_1) \rangle$ contains $e(H_2)$ and $e(x_2)$, $\Sigma_2 \subseteq \langle \alpha_1, \alpha_2, e(x_1) \rangle$ (similar reasoning as above). Hence, $\Sigma = \langle \Sigma_1, \Sigma_2 \rangle \subseteq \langle \alpha_1, \alpha_2, e(x_1) \rangle$. But this is impossible, since $\dim(\Sigma) = 2^n - 1$ and $\dim\langle \alpha_1, \alpha_2, e(x_1) \rangle \leq 2^n - 2$. So, the claim is correct.

By the previous claim and Proposition 3.1, it readily follows that there is a unique locally singular hyperplane of Δ containing H_1, H_2 and $x \in L$, namely the locally singular hyperplane of Δ arising from the hyperplane $\langle \alpha_1, \alpha_2, e(x) \rangle$ of Σ. ■

Lemma 3.5 *Let H be a locally singular hyperplane of Δ. Then the set of deep points (with respect to H) is a subspace of Δ.*

Proof. Let x_1 and x_2 be two distinct collinear points of H which are deep with respect to H, and let x_3 denote a third point of the line x_1x_2. If Q is a quad through the line x_1x_2, then $Q \subseteq H$, since $x_1^\perp \cap Q \subseteq H$ and $x_2^\perp \cap Q \subseteq H$. Since this holds for every quad Q through x_1x_2, also the point x_3 is deep with respect to H. ■

Lemma 3.6 *If H_1 and H_2 are two distinct locally singular hyperplanes of Δ, then there exists a point in $H_1 \setminus H_2$ which is not deep with respect to H_1.*

Proof. Obviously, there exists a point $u \in H_1 \setminus H_2$ (recall that H_1 and H_2 are maximal subspaces) and a point $v \in H_1$ which is not deep with respect to H_1 (since H_1 is a proper subspace). We choose such points u and v with $\mathrm{d}(u, v)$ as small as possible. If $\mathrm{d}(u, v) = 0$, then we are done. So, suppose $\mathrm{d}(u, v) \geq 1$. Then u is deep with respect to H_1 and $v \in H_1 \cap H_2$. Let L_v denote a line through v contained in $H_1 \cap \langle u, v \rangle$. Notice that if $\mathrm{d}(u, v) = 1$, then $L_v = uv$. If $\mathrm{d}(u, v) \geq 2$, then such a line exists in any quad of $\langle u, v \rangle$ through v (recall that H_1 is locally singular). Let v' denote the point of L_v nearest to u and let L_u denote a line of $\langle u, v \rangle$ through u not contained in $\langle u, v' \rangle$. Then every point of $L_u \subseteq H_1$ has distance $\mathrm{d}(u, v) - 1$ from L_v. Now, precisely one point of L_u belongs to H_2, and by Lemma 3.5, at most one point of L_v is deep with respect to H_1. Hence, there exist points $u_1 \in L_u$ and $v_1 \in L_v$ satisfying the following properties:

- $u_1 \in H_1 \setminus H_2$;
- $v_1 \in H_1$ and v_1 is not deep with respect to H_1;
- $\mathrm{d}(u_1, v_1) = \mathrm{d}(u, v) - 1$.

This contradicts the minimality of $\mathrm{d}(u,v)$. Hence, the lemma holds. ∎

Now, let H_1 and H_2 be two distinct locally singular hyperplanes of Δ. Let Γ_{H_1,H_2} be the graph with vertices the points of $P \setminus (H_1 \cup H_2)$, with two distinct vertices adjacent whenever either (i) or (ii) below holds:

(i)
- $\mathrm{d}(x,y) = 1$;
- the line xy meets $H_1 \cap H_2$.

(ii)
- $\mathrm{d}(x,y) = 2$;
- $\langle x,y \rangle \cap H_1 \cap H_2$ is a line L;
- $\pi_L(x) = \pi_L(y)$.

Let $\mathcal{V}$ denote the set of all connected components of Γ_{H_1,H_2}, and define

$$[H_1, H_2] := \{H_1, H_2\} \cup \{V \cup (H_1 \cap H_2) \,|\, V \in \mathcal{V}\}.$$

Notice that in [5] there was given a slightly different but equivalent definition of the set $\mathcal{V}$.

Lemma 3.7 (Proposition 2.2 of [5]) *If H is a locally singular hyperplane of Δ such that $H \cap H_1 = H \cap H_2 = H_1 \cap H_2$, then $H \in [H_1, H_2]$.*

Lemma 3.8 *$[H_1, H_2]$ is the unique pencil of locally singular hyperplanes of Δ containing H_1 and H_2.*

Proof. Let α_i, $i \in \{1,2\}$, denote the hyperplane of Σ giving rise to H_i. Let $\mathcal{W}$ denote the set of all locally singular hyperplanes of Δ arising from a hyperplane of Σ through $\alpha_1 \cap \alpha_2$. Then $\mathcal{W}$ is a pencil of locally singular hyperplanes. By Lemma 3.7, $\mathcal{W} = [H_1, H_2]$. From Lemma 3.7, it is also clear that $[H_1, H_2]$ is the unique pencil of locally singular hyperplanes of Δ containing H_1 and H_2. ∎

The set $\mathcal{H}$ of all locally singular hyperplanes of Δ carries the structure of a projective space isomorphic to $\mathrm{PG}(2^n - 1, \mathbb{K})$ if we take the sets $[H_1, H_2]$, $H_1, H_2 \in \mathcal{H}$ and $H_1 \neq H_2$, as lines. (Recall that there exists a bijective correspondence between the elements of $\mathcal{H}$ and the points of the projective space Σ^*, dual of Σ.) If $H_1, H_2, \ldots, H_k$ are $k \geq 1$ elements of $\mathcal{H}$, then we denote by $[H_1, H_2, \ldots, H_k]$ the subspace of the projective space $\mathcal{H}$ generated by $H_1, H_2, \ldots, H_k$.

Lemma 3.9 *There exist 2^n singular hyperplanes in $\mathcal{H}$ which generate $\mathcal{H}$.*

Proof. We must show that there exist 2^n singular hyperplanes $H_1, \ldots, H_{2^n}$ in $\mathcal{H}$ such that $\langle e(H_1)\rangle \cap \langle e(H_2)\rangle \cap \cdots \cap \langle e(H_{2^n})\rangle = \emptyset$. But this follows immediately from the fact that the spin-embedding of Δ is the so-called minimal full polarized embedding of Δ, see Cardinali, De Bruyn and Pasini [3]. ∎

4 Proof of Theorem 1.1

Let f be an isometric embedding of the dual polar space $\Delta := DQ(2n, \mathbb{K})$ into the dual polar space $\Delta' := DQ(2n, \mathbb{K}')$, $n \geq 2$. Let P and P' denote the point sets of Δ and Δ', respectively.

Lemma 4.1 *For every locally singular hyperplane H of Δ, there is at most one locally singular hyperplane H' of Δ' such that $f(H) = H' \cap f(P)$.*

Proof. We will prove this lemma by induction on n. We will use the same notations as in Section 2.

Suppose $n = 2$. Then H is a singular hyperplane of Δ. Let x denote the deepest point of H and let L_1 and L_2 denote two distinct lines of Δ through x. If H' is a locally singular hyperplane of Δ' such that $f(H) = H' \cap f(P)$, then $f(L_1), f(L_2) \subseteq H'$. Hence, H' coincides with the singular hyperplane of Δ' with deepest point $f(x)$.

Suppose $n \geq 3$. Let M_1, M_2 and M_3 denote three mutually disjoint maxes of Δ. By Lemma 3.2, at most one of M_1, M_2, M_3 is contained in H. So, without loss of generality, we may suppose that M_1 and M_2 are not contained in H. Let H_i, $i \in \{1, 2\}$, be the locally singular hyperplane $M_i \cap H$ of M_i. By Lemma 3.3, there is a point $x_1 \in M_1 \setminus (H_1 \cup \pi_{M_1}(H_2))$. Put $x_2 := \pi_{M_2}(x_1)$. Then $x_2 \notin H_2$. Let L be the line x_1x_2 and let x_3 be the unique point of L contained in H. Notice $x_3 \notin \{x_1, x_2\}$. By Proposition 2.4, $M_1' := (M_1)_f$ and $M_2' := (M_2)_f$ are two disjoint maxes of Δ' and L_f is a line of Δ' intersecting M_1' and M_2' in the respective points $f(x_1)$ and $f(x_2)$.

Suppose now that H' is a locally singular hyperplane of Δ' such that $f(H) = H' \cap f(P)$. We will show that H' is uniquely determined by H. Since $x_3 \in H$, $f(x_3) \in H'$. By Proposition 2.4, $f(P) \cap M_i' = f(M_i)$. So, we obtain

$$f(H) \cap M_i' \quad = \quad H' \cap M_i' \cap f(P)$$

$$\begin{aligned} f(H) \cap (f(P) \cap M_i') &= (H' \cap M_i') \cap (M_i' \cap f(P)) \\ f(H) \cap f(M_i) &= (H' \cap M_i') \cap f(M_i) \\ f(H_i) &= (H' \cap M_i') \cap f(M_i). \end{aligned}$$

By the induction hypothesis, $H' \cap M_i'$ is the unique locally singular hyperplane G_i' of M_i' such that $f(H_i) = G_i' \cap f(M_i)$. Since $x_i \notin H_i$, $f(x_i) \notin G_i'$. From Lemma 3.4, it now readily follows that H' is the unique locally singular hyperplane of Δ' containing G_1', G_2' and $f(x_3)$. So, H' is uniquely determined by H. ∎

Lemma 4.2 *Let H_1 and H_2 be two distinct locally singular hyperplanes of Δ. If there exist locally singular hyperplanes H_1' and H_2' in Δ' such that $f(H_1) = f(P) \cap H_1'$ and $f(H_2) = f(P) \cap H_2'$, then for every locally singular hyperplane H of $[H_1, H_2]$, there exists a locally singular hyperplane H' of $[H_1', H_2']$ such that $f(H) = f(P) \cap H'$.*

Proof. Remark that $H_1' \neq H_2'$ since $H_1 \neq H_2$. We may suppose that $H_1 \neq H \neq H_2$. Let x denote an arbitrary point of $H \setminus (H_1 \cap H_2)$. Since $x \notin H_1 \cup H_2$, $f(x) \notin H_1' \cup H_2'$. Let H' denote the unique hyperplane of $[H_1', H_2']$ containing $f(x)$.

We will show that $f(H) \subseteq f(P) \cap H'$. We have $f(H_1 \cap H_2) = f(H_1) \cap f(H_2) = f(P) \cap H_1' \cap H_2' \subseteq f(P) \cap H'$. So, we still must show that $f(H \setminus (H_1 \cap H_2)) \subseteq f(P) \cap H'$. Let Γ_{H_1,H_2} be the graph with vertex set $P \setminus (H_1 \cup H_2)$ as defined in Section 3. We show the following: if $y_1, y_2 \in H \setminus (H_1 \cap H_2)$ are adjacent vertices of Γ_{H_1,H_2} such that $f(y_1) \in f(P) \cap H'$, then also $f(y_2) \in f(P) \cap H'$. The claim then follows from Lemma 3.7 and the fact that $f(x) \in f(P) \cap H'$.

Suppose first that $y_1 y_2$ meets $H_1 \cap H_2$ in a point y_3. The line $f(y_1)f(y_2)$ of Δ' contains the point $f(y_3) \in f(H_1 \cap H_2) \subseteq H'$. Since $f(y_1) \in H'$, also $f(y_2) \in H'$.

Suppose next that the following holds: $\mathrm{d}(y_1, y_2) = 2$; $\langle y_1, y_2 \rangle \cap H_1 \cap H_2$ is a line L; $\pi_L(y_1) = \pi_L(y_2)$. Put $Q := \langle y_1, y_2 \rangle$ and $x_3 := \pi_L(y_1) = \pi_L(y_2)$. Let x_i, $i \in \{1, 2\}$, denote the deepest point of the singular hyperplane $Q \cap H_i$ of Q. Then L contains the points x_1, x_2 and x_3. The quad Q_f contains the line L_f which itself contains the points $f(x_1)$, $f(x_2)$ and $f(x_3)$. Since $f(H_i) = f(P) \cap H_i'$, $H_i' \cap Q_f$ is the singular hyperplane of Q_f with deepest point $f(x_i)$. Since $H' \in [H_1', H_2']$, $H' \cap Q_f$ is a singular hyperplane whose deepest point lies on L_f. (Notice that the set of all singular hyperplanes of Q_f whose deepest points lie on L_f is the unique pencil of locally singular hyperplanes of Q_f containing $f(x_1)^\perp \cap Q_f$ and $f(x_2)^\perp \cap Q_f$.) Since $f(y_1) \in H'$, the deepest point of $H' \cap Q_f$ coincides with $\pi_{L_f}(f(y_1)) = f(x_3)$. Now, $f(y_2)$ is collinear with $f(x_3)$. Hence, $f(y_2) \in H'$. This was what we needed to show.

We will now show that $f(H) = f(P) \cap H'$. Suppose $f(x')$ is a point of $f(P) \cap H'$ not contained in $f(H)$. Then x' is a point of $P \setminus (H_1 \cup H_2 \cup H)$. Let G denote the unique element of $[H_1, H_2]$ containing x'. Since $f(x') \subseteq H'$, $f(G) \subseteq H'$ by the above reasoning. Now, by Lemma 3.6, there exists a point $u \in H_1 \setminus H_2$ which is not deep with respect to H_1. Let L denote a line through u which is not contained in H_1. Put $\{v\} = L \cap H_2$, $\{w\} = L \cap H$ and $\{w'\} = L \cap G$. Since $f(w), f(w') \in H'$, $f(z) \in H'$ for every $z \in L$. In particular, $f(u) \in H'$. This implies $f(u) \in H' \cap H_1' \cap f(P) = H_1' \cap H_2' \cap f(P) = f(H_1 \cap H_2)$, contradicting $u \in H_1 \setminus H_2$. Hence, $f(H) = f(P) \cap H'$ as claimed. ■

Lemma 4.3 *For every locally singular hyperplane H of Δ, there exists a hyperplane H' of Δ' such that $f(H) = f(P) \cap H'$.*

Proof. By Lemmas 3.9 and 4.2, it suffices to prove the lemma in the case that H is a singular hyperplane of Δ. So, suppose that H is singular and that x is the deepest point of H. Let H' denote the singular hyperplane of Δ' with deepest point $f(x)$. Since f is an isometric embedding, we necessarily have $f(H) = f(P) \cap H'$. This proves the lemma. ■

Theorem 1.1 is a consequence of Lemmas 4.1 and 4.3.

5 Proof of Theorem 1.2

Let f be an isometric embedding of $\Delta = DQ(2n, \mathbb{K})$ into $\Delta' = DQ(2n, \mathbb{K}')$. Let $\mathcal{H}$ denote the set of all locally singular hyperplanes of Δ and let $\mathcal{H}'$ denote the set of all locally singular hyperplanes of Δ'. For every hyperplane H of $\mathcal{H}$, let $\theta(H)$ denote the unique hyperplane of $\mathcal{H}'$ for which $f(H) = f(P) \cap \theta(H)$. As explained above, the sets $\mathcal{H}$ and $\mathcal{H}'$ can be given the structure of $(2^n - 1)$-dimensional projective spaces. Obviously, the map θ defines an injection from the point-set of $\mathcal{H}$ to the point set of $\mathcal{H}'$. By Lemma 4.2, θ maps lines of $\mathcal{H}$ to subsets of lines of $\mathcal{H}'$. Hence, we have

Lemma 5.1 *Let $H_1, H_2, \ldots, H_k$ be elements of $\mathcal{H}$. If $H \in [H_1, H_2, \ldots, H_k]$, then $\theta(H) \in [\theta(H_1), \theta(H_2), \ldots, \theta(H_k)]$.*

Definition. A nonempty set X of points of a thick dual polar space $\widetilde{\Delta}$ is called *scattered* if $\bigcap_{x \in X} H_x = \emptyset$. Here, H_x denotes the singular hyperplane of $\widetilde{\Delta}$ with deepest point x. A scattered set X of points is called *minimal* if no proper subset of X is scattered. By De Bruyn and Pasini [8], every dual polar space of rank n has minimal scattered sets of size 2^n.

Lemma 5.2 $\langle \theta(\mathcal{H}) \rangle = \mathcal{H}'$.

Proof. Let $x_1, x_2, \ldots, x_{2^n}$ be a set of 2^n points in Δ which form a minimal scattered set of points. Let H_{x_i}, $i \in \{1, \ldots, 2^n\}$, be the singular hyperplane of Δ with deepest point x_i, and let H'_{x_i} denote the singular hyperplane of Δ' with deepest point $f(x_i)$. Then $\theta(H_{x_i}) = H'_{x_i}$. Now, since $\{x_1, x_2, \ldots, x_{2^n}\}$ is a minimal scattered set of points,

$$H_{x_1} \cap H_{x_2} \cap \cdots \cap H_{x_{i+1}} \subsetneqq H_{x_1} \cap H_{x_2} \cap \cdots \cap H_{x_i}$$

for every $i \in \{1, \ldots, 2^n - 1\}$. Now, since $f(H_{x_i}) = f(P) \cap H'_{x_i}$ for every $i \in \{1, \ldots, 2^n\}$, we have

$$H'_{x_1} \cap H'_{x_2} \cap \cdots \cap H'_{x_{i+1}} \subsetneqq H'_{x_1} \cap H'_{x_2} \cap \cdots \cap H'_{x_i}$$

for every $i \in \{1, \ldots, 2^n - 1\}$. If $y \in H'_{x_1} \cap H'_{x_2} \cap \cdots \cap H'_{x_i}$, then y belongs to every hyperplane of $[H'_{x_1}, H'_{x_2}, \ldots, H'_{x_i}]$. Hence, $H'_{x_{i+1}} \notin [H'_{x_1}, H'_{x_2}, \ldots, H'_{x_i}]$ for every $i \in \{1, \ldots, 2^n - 1\}$. So, the points H'_{x_1}, H'_{x_2}, ..., $H'_{x_{2^n}}$ of $\mathcal{H}'$ are linearly independent. It follows that $[H'_{x_1}, \ldots, H'_{x_{2^n}}] = \mathcal{H}'$, which implies that $\langle \theta(\mathcal{H}) \rangle = \mathcal{H}'$. ■

Lemma 5.3 *If $\{H_1, H_2, \ldots, H_k\}$ is a linearly independent set of points of $\mathcal{H}$, then $\{\theta(H_1), \theta(H_2), \ldots, \theta(H_k)\}$ is a linearly independent set of points of $\mathcal{H}'$.*

Proof. Complete $H_1, H_2, \ldots, H_k$ to a generating set $H_1, \ldots, H_k, \ldots, H_{2^n}$ of $\mathcal{H}$. By Lemmas 5.1 and 5.2, $\mathcal{H}' = \langle \theta(\mathcal{H}) \rangle = \langle \theta(H_1), \theta(H_2), \ldots, \theta(H_{2^n}) \rangle$. It follows that $\theta(H_1), \theta(H_2), \ldots, \theta(H_{2^n})$ are linearly independent. In particular, $\theta(H_1), \theta(H_2), \ldots, \theta(H_k)$ are linearly independent. ■

Definition. For every subspace α of $\mathcal{H}$, let $\theta(\alpha)$ be the subspace of $\mathcal{H}'$ generated by all points $\theta(H)$, $H \in \alpha$. Then $\dim(\alpha) = \dim(\theta(\alpha))$ by Lemmas 5.1 and 5.3.

Corollary 5.4 *The points $\theta(H)$, $H \in \mathcal{H}$, define a subgeometry of $\mathcal{H}'$ isomorphic to $\mathcal{H}$.*

For every point x (respectively line L) of Δ, let V_x (respectively V_L) denote the set of all hyperplanes of $\mathcal{H}$ containing the point x (respectively the line L) of Δ. Then V_x is a hyperplane of $\mathcal{H}$ and V_L is a hyperplane of V_y for every point y of L. So, V_L is a $(2^n - 3)$-dimensional subspace of $\mathcal{H}$.

Similarly, for every point x (respectively line L) of Δ', let V'_x (respectively V'_L) denote the set of all hyperplanes of $\mathcal{H}'$ containing x (respectively L). Then V'_x is a hyperplane of $\mathcal{H}'$ and V'_L is a $(2^n - 3)$-dimensional subspace of $\mathcal{H}'$.

Lemma 5.5 *Let x be a point of Δ and let L be a line of Δ. Then $\theta(V_x) = V'_{f(x)}$ and $\theta(V_L) = V'_{L_f}$.*

Proof. Obviously, $\theta(V_x) \subseteq V'_{f(x)}$. Since both subspaces are $(2^n - 2)$-dimensional, $\theta(V_x) = V'_{f(x)}$. In a similar way, one shows that $\theta(V_L) = V'_{L_f}$. ■

Let $\mathcal{H}^*$ and $\mathcal{H}'^*$ denote the dual projective spaces of $\mathcal{H}$ and $\mathcal{H}'$, respectively. The points of $\mathcal{H}^*$ are mapped by θ to a subgeometry of $\mathcal{H}'^*$ isomorphic to $\mathcal{H}^*$.

The map $e_1 : P \to \mathcal{H}^*; x \mapsto V_x$ defines a full embedding of Δ into the projective space $\mathcal{H}^*$, isomorphic to the spin-embedding of Δ. The map $e_2 : P' \to \mathcal{H}'^*; x \mapsto V'_x$ defines a full embedding of Δ' into the projective space $\mathcal{H}'^*$, isomorphic to the spin-embedding of Δ'.

For every point x of Δ, we have $e_2 \circ f(x) = V'_{f(x)} = \theta(V_x) = \theta(e_1(x))$. Theorem 1.2 is now obvious.

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